

The endomorphisms of Grassmann graphs^{*}

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Abstract

A graph G is a core if every endomorphism of G is an automorphism. A graph is called a pseudo-core if every its endomorphism is either an automorphism or a colouring. Suppose that $J_q(n, m)$ is a Grassmann graph over a finite field with q elements. We show that every Grassmann graph is a pseudo-core. Moreover, $J_2(4, 2)$ is not a core and $J_q(2k + 1, 2)$ ($k \geq 2$) is a core. Further, if m and $n - m + 1$ are not relatively prime, then $J_q(n, m)$ is a core when q is a sufficiently large integer.

Key words Grassmann graph, core, pseudo-core, endomorphism, maximal clique

1 Introduction

Throughout this paper, all graphs are finite undirected graphs without loops or multiple edges. For a graph G , we let $V(G)$ denote the vertex set of G . If xy is an edge of G , x and y are said to be *adjacent*, denoted by $x \sim y$. Let G and H be two graphs. A *homomorphism* φ from G to H is a mapping $\varphi : V(G) \rightarrow V(H)$ such that $\varphi(x) \sim \varphi(y)$ whenever $x \sim y$. If H is the complete graph K_r , then φ is a *r-colouring* of G (*colouring* for short). An *isomorphism* from G to H is a bijection $\varphi : V(G) \rightarrow V(H)$ such that $x \sim y \Leftrightarrow \varphi(x) \sim \varphi(y)$. Graphs G and H are called isomorphic if there is an isomorphism from G to H , and denoted by $G \cong H$. A homomorphism (resp. isomorphism) from G to itself is called an *endomorphism* (resp. *automorphism*) of G .

Recall that a graph G is a *core* if every endomorphism of G is an automorphism. A subgraph H of G is a *core* of G if it is a core and there exists a homomorphism from G to H . Every graph has a core, which is an induced subgraph and is unique up to isomorphism [6]. A graph is called *core-complete* if it is a core or its core is complete.

A graph G is called a *pseudo-core* if every endomorphism of G is either an automorphism or a colouring. Every core is a pseudo-core. Any pseudo-core is core-complete but not vice versa. For more information, see [2, 7, 10].

For a graph G , an important and difficult problem is to distinguish whether G is a core [2, 6, 7, 8, 12, 17]. If G is not a core or we don't know whether it is a core, then we need to judge whether it is a pseudo-core because the concept of pseudo-core is the most close to the core. Recently, Godsil and Royle [7] discussed some properties of the pseudo-core of a graph. Cameron and Kazanidis [2] discussed the core-complete graph and the cores of symmetric graphs. The literature [11] showed that every bilinear forms graph is a pseudo-core which is not a core. One of the latest result is that the literature [10] proved

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that every alternating forms graph is a pseudo-core. Moreover, Orel [14, 15] proved that each symmetric bilinear forms graph (whose diameter is greater than 2) is a core and each Hermitian forms graph is a core.

Suppose that \mathbb{F}_q is the finite field with q elements, where q is a power of a prime. Let V be an n -dimensional row vector space over \mathbb{F}_q and let $\begin{bmatrix} V \\ m \end{bmatrix}$ be the set of all m -dimensional subspaces of V . The *Grassmann graph* $J_q(n, m)$ has the vertex set $\begin{bmatrix} V \\ m \end{bmatrix}$, and two vertices are adjacent if their intersection is of dimension $m-1$. If $m = 1$, we have a complete graph and hence it is a core. Since $J_q(n, m) \cong J_q(n, n-m)$, we always assume that $4 \leq 2m \leq n$ in our discussion unless specified otherwise. The number of vertices of $J_q(n, m)$ is the Gaussian binomial coefficient:

$$\begin{bmatrix} n \\ m \end{bmatrix} = \prod_{i=1}^m \frac{q^{n+1-i} - 1}{q^i - 1}. \quad (1)$$

For $J_q(n, m)$, the distance of two vertices X and Y is $d(X, Y) := m - \dim(X \cap Y)$. Any Grassmann graph is distance-transitive [1, Theorem 9.3.3] and connected. By [7, Corollary 4.2], every distance-transitive graph is core-complete, thus every Grassmann graph is core-complete. The Grassmann graph plays an important role in geometry, graph theory, association schemes and coding theory.

Recall that an *independent set* of a graph G is a set of vertices that induces an empty subgraph. The size of the largest independent set is called the *independence number* of G , denoted by $\alpha(G)$. The *chromatic number* $\chi(G)$ of G is the least value of k for which G can be k -colouring. A *clique* of a graph G is a complete subgraph of G . A clique C is maximal if there is no clique of G which properly contains C as a subset. A *maximum clique* of G is a clique with the maximum size. The *clique number* of G is the number of vertices in a maximum clique, denoted by $\omega(G)$.

By [7, p.273], if G is a distance-transitive graph and $\chi(G) > \omega(G)$, then G is a core. Unluckily, applying the eigenvalues or the known results of graph theory for Grassmann graph, to prove the inequality $\chi(G) > \omega(G)$ is difficult. Thus, it is a difficult problem to verify a Grassmann graph being a core. However, there are some Grassmann graphs which are not cores (see Section 4). Therefore, we need to judge whether a Grassmann graph is a pseudo-core. So far, this is an open problem. We solve this problem as follows:

Theorem 1.1 *Every Grassmann graph $J_q(n, m)$ is a pseudo-core.*

The paper is organized as follows. In section 2, we give some properties of the maximal cliques of Grassmann graphs. In section 3, we shall prove Theorem 1.1. In Section 4, we discuss cores on Grassmann graphs. We shall show that $J_2(4, 2)$ is not a core, $J_q(2k+1, 2)$ ($k \geq 2$) is a core. Moreover, if m and $n-m+1$ are not relatively prime, then $J_q(n, m)$ is a core when q is a sufficiently large integer.

2 Maximal cliques of Grassmann graph

In this section we shall discuss some properties of the maximal cliques of Grassmann graphs.

We will denote by $|X|$ the cardinal number of a set X . Suppose that V is an n -dimensional row vector space over \mathbb{F}_q . For two vector subspaces S and T of V , the *join* $S \vee T$ is the minimal dimensional vector subspace containing S and T . We have the dimensional formula (cf. [9, Lemma 2.1] or [18]):

$$\dim(S \vee T) = \dim(S) + \dim(T) - \dim(S \cap T). \quad (2)$$

Throughout this section, suppose that $4 \leq 2m \leq n$. For every $(m-1)$ -dimensional subspace P of V , let $[P]_m$ denote the set of all m -dimensional subspaces containing P , which is called a *star*. For every

$(m + 1)$ -dimensional subspace Q of V , let $\langle Q \rangle_m$ denote the set of all m -dimensional subspaces of Q , which is called a *top*. By [5], every maximal clique of $J_q(n, m)$ is a star or a top. For more information, see [16].

By [18, Corollary 1.9],

$$|[\langle P \rangle_m]| = \frac{q^{n-m+1} - 1}{q - 1}, \quad |\langle Q \rangle_m| = \frac{q^{m+1} - 1}{q - 1}. \quad (3)$$

If $n > 2m$, every maximum clique of $J_q(n, m)$ is a star. If $n = 2m$, every maximal clique of $J_q(n, m)$ is a maximum clique. By (3) we have

$$\omega(J_q(n, m)) = \begin{bmatrix} n-m+1 \\ 1 \end{bmatrix} \text{ if } n \geq 2m, \text{ or } \omega(J_q(n, m)) = \begin{bmatrix} m+1 \\ 1 \end{bmatrix} \text{ if } n < 2m. \quad (4)$$

Since $n \geq 2m$, we have

$$|[\langle P \rangle_m]| \geq |\langle Q \rangle_m|, \text{ and } |[\langle P \rangle_m]| > |\langle Q \rangle_m| \text{ if } n > 2m. \quad (5)$$

Lemma 2.1 *If $[\langle P \rangle_m] \cap \langle Q \rangle_m \neq \emptyset$, then the size of $[\langle P \rangle_m] \cap \langle Q \rangle_m$ is $q + 1$.*

Proof. Since $[\langle P \rangle_m] \cap \langle Q \rangle_m \neq \emptyset$, one gets $P \subseteq Q$. It follows that $[\langle P \rangle_m] \cap \langle Q \rangle_m$ consists of all m -dimensional subspaces containing P in Q . By [18, Corollary 1.9], the desired result follows. \square

Lemma 2.2 ([9, Corollary 4.4]) *Let \mathcal{M}_1 and \mathcal{M}_2 be two distinct stars (tops). Then $|\mathcal{M}_1 \cap \mathcal{M}_2| \leq 1$.*

Lemma 2.3 *Suppose $[A]_m \neq [B]_m$. Then $[A]_m \cap [B]_m \neq \emptyset$ if and only if $\dim(A \cap B) = m - 2$. In this case, $[A]_m \cap [B]_m = \{A \vee B\}$.*

Proof. Since $\dim(A) = \dim(B) = m - 1$ and $A \neq B$, one gets $\dim(A \vee B) \geq m$. If $[A]_m \cap [B]_m \neq \emptyset$, then by Lemma 2.2, there exists a vertex C of $J_q(n, m)$ such that $\{C\} = [A]_m \cap [B]_m$. It follows from (2) and $A, B \subset C$ that $C = A \vee B$ and $\dim(A \cap B) = m - 2$. Conversely, if $\dim(A \cap B) = m - 2$, then Lemma 2.2 and (2) imply that $C := A \vee B$ is a vertex of $J_q(n, m)$ and hence $\{C\} = [A]_m \cap [B]_m$. \square

Lemma 2.4 *Suppose $\langle P \rangle_m \neq \langle Q \rangle_m$. Then $\langle P \rangle_m \cap \langle Q \rangle_m \neq \emptyset$ if and only if $\dim(P \cap Q) = m$. In this case, $\langle P \rangle_m \cap \langle Q \rangle_m = \{P \cap Q\}$.*

Proof. By $\dim(P) = \dim(Q) = m + 1$ and $P \neq Q$, we have $\dim(P \cap Q) \leq m$. If $\langle P \rangle_m \cap \langle Q \rangle_m \neq \emptyset$, then Lemma 2.2 implies that there exists a vertex C of $J_q(n, m)$ such that $\{C\} = \langle P \rangle_m \cap \langle Q \rangle_m$. Since $C \subset P \cap Q$, we get that $C = P \cap Q$ and $\dim(P \cap Q) = m$. Conversely, if $\dim(P \cap Q) = m$, then by $P \cap Q \in \langle P \rangle_m \cap \langle Q \rangle_m$ and Lemma 2.2, we have $\{P \cap Q\} = \langle P \rangle_m \cap \langle Q \rangle_m$. \square

In the following, let φ be an endomorphism of $J_q(n, m)$ and let $\text{Im}(\varphi)$ be the image of φ .

Lemma 2.5 *If \mathcal{M} is a maximal clique, then there exists a unique maximal clique containing $\varphi(\mathcal{M})$.*

Proof. Suppose there exist two distinct maximal cliques \mathcal{M}' and \mathcal{M}'' containing $\varphi(\mathcal{M})$. Then $\varphi(\mathcal{M}) \subseteq \mathcal{M}' \cap \mathcal{M}''$. By Lemmas 2.1 and 2.2, $|\mathcal{M}' \cap \mathcal{M}''| \leq q + 1$. Since $|\mathcal{M}| = |\varphi(\mathcal{M})|$, by (3) we have $|\varphi(\mathcal{M})| > q + 1$, a contradiction. \square

Lemma 2.6 *Let \mathcal{M} be a star and \mathcal{N} be a top such that $|\varphi(\mathcal{M}) \cap \varphi(\mathcal{N})| > q + 1$. Then $\varphi(\mathcal{N}) \subseteq \varphi(\mathcal{M})$.*

Proof. Let \mathcal{N}' be the maximal clique containing $\varphi(\mathcal{N})$. Then $|\varphi(\mathcal{M}) \cap \mathcal{N}'| > q+1$. One gets $\varphi(\mathcal{M}) = \mathcal{N}'$ by Lemmas 2.1 and 2.2. \square

Lemma 2.7 *Suppose there exist two distinct stars $[A]_m$ and $[B]_m$ such that*

$$[A]_m \cap [B]_m = \{X\}, \quad \varphi([A]_m) = \varphi([B]_m).$$

If $\varphi([A]_m)$ is a star, then φ is a colouring of $J_q(n, m)$.

Proof. Write $\mathcal{M} := \varphi([A]_m)$. Then $\varphi([B]_m) = \mathcal{M}$ and $\varphi(X) \in \mathcal{M}$. Since the restriction mapping of φ on a maximal clique is injective and (5), it is easy to see that \mathcal{M} is a star. If $\text{Im}(\varphi) = \mathcal{M}$, then φ is a colouring of $J_q(n, m)$. Now we prove $\text{Im}(\varphi) = \mathcal{M}$ as follows. Suppose that Y is any vertex with $Y \sim X$. Since $G := J_q(n, m)$ is connected, it suffices to show that there exist two distinct stars $[C]_m$ and $[D]_m$ such that

$$\{Y\} = [C]_m \cap [D]_m \quad \text{and} \quad \varphi([C]_m) = \varphi([D]_m) = \mathcal{M}.$$

In fact, if we can prove this point, then we can imply that $\varphi(Z) \in \mathcal{M}$ for all $Z \in V(G)$. We prove it as follows.

Since $X \in \langle X \vee Y \rangle_m \cap [A]_m \cap [B]_m$, using Lemma 2.2 we get $|\langle X \vee Y \rangle_m \cap [A]_m \cap [B]_m| = 1$. By Lemma 2.1 we obtain

$$|\langle X \vee Y \rangle_m \cap [A]_m| = |\langle X \vee Y \rangle_m \cap [B]_m| = q+1.$$

It follows that

$$|\langle X \vee Y \rangle_m \cap ([A]_m \cup [B]_m)| = 2q+1.$$

Observe that

$$\varphi(\langle X \vee Y \rangle_m \cap ([A]_m \cup [B]_m)) \subseteq \varphi(\langle X \vee Y \rangle_m) \cap \varphi([A]_m \cup [B]_m) \subseteq \varphi(\langle X \vee Y \rangle_m) \cap \mathcal{M}.$$

Since the restriction of φ on a clique is injective, one gets

$$|\varphi(\langle X \vee Y \rangle_m) \cap \mathcal{M}| \geq 2q+1 > q+1.$$

Thus, Lemma 2.6 implies that

$$\varphi(\langle X \vee Y \rangle_m) \subseteq \mathcal{M}. \tag{6}$$

So $\varphi(Y) \in \mathcal{M}$. Write $C := X \cap Y$. Since every vertex of $[C]_m \setminus \{X\}$ is adjacent to X , by our claim we have $\varphi([C]_m) = \mathcal{M}$.

Pick a vertex Z such that $Z \sim Y$ and the distance from X is 2. Write $D = Y \cap Z$. Since $Y \in [D]_m \cap \langle X \vee Y \rangle_m$, by Lemma 2.1 we have $|[D]_m \cap \langle X \vee Y \rangle_m| = q+1$. It follows from (6) that $|\varphi([D]_m) \cap \mathcal{M}| \geq q+1$. Thus Lemma 2.2 implies that $\varphi([D]_m) = \mathcal{M}$. Since $\{Y\} = [C]_m \cap [D]_m$, $[C]_m$ and $[D]_m$ are the desired stars. \square

3 Proof of Theorem 1.1

For the proof of Theorem 1.1, we only need to consider the case $4 \leq 2m \leq n$. We divide the proof of Theorem 1.1 into two cases: $n > 2m$ and $n = 2m$.

Lemma 3.1 *If $n > 2m$, then every Grassmann graph $J_q(n, m)$ is a pseudo-core.*

Proof. Suppose that $n > 2m \geq 4$. Then by (5), every maximum clique of $J_q(n, m)$ is a star. Let φ be an endomorphism of $J_q(n, m)$. Then the restriction of φ on any clique is injective, so φ transfers stars to stars.

Suppose φ is not a colouring. It suffices to show that φ is an automorphism. Write $G_r := J_q(n, r)$, where $1 \leq r \leq m-1$. By Lemma 2.7, the images under φ of any two distinct and intersecting stars are distinct. Hence by Lemma 2.3, φ induces an endomorphism φ_{m-1} of G_{m-1} such that

$$\varphi([A]_m) = [\varphi_{m-1}(A)]_m.$$

Let X be any vertex of $J_q(n, m)$. Then there exist two vertices X' and X'' of G_{m-1} such that $X = X' \vee X''$. Then $[X']_m \cap [X'']_m = \{X\}$ and $\varphi(X) \in \varphi([X']_m) \cap \varphi([X'']_m)$. Since φ is not a colouring, by Lemma 2.7 $\varphi([X']_m)$ and $\varphi([X'']_m)$ are two distinct stars. By Lemma 2.2, $[\varphi_{m-1}(X')]_m \cap [\varphi_{m-1}(X'')]_m = \{\varphi(X)\}$. Thus Lemma 2.3 implies that

$$\varphi(X) = \varphi_{m-1}(X') \vee \varphi_{m-1}(X''). \quad (7)$$

When $m = 2$, G_1 is a complete graph, hence it is a core. We next show that φ_{m-1} is not a colouring of G_{m-1} for $m \geq 3$. For any two vertices A_1 and A_3 of G_{m-1} at distance 2, we claim that

$$\varphi_{m-1}(A_1) \neq \varphi_{m-1}(A_3).$$

There exists an $A_2 \in V(G_{m-1})$ such that $A_1 \sim A_2 \sim A_3$. Write $Y_1 := A_1 \vee A_2$ and $Y_2 := A_2 \vee A_3$. Then $Y_1 \sim Y_2$, so $\varphi(Y_1) \neq \varphi(Y_2)$. By (7),

$$\varphi(Y_1) = \varphi_{m-1}(A_1) \vee \varphi_{m-1}(A_2), \quad \varphi(Y_2) = \varphi_{m-1}(A_2) \vee \varphi_{m-1}(A_3).$$

Thus our claim is valid. Otherwise, one has $\varphi(Y_1) = \varphi(Y_2)$, a contradiction.

Pick a star \mathcal{N} of G_{m-1} . Since the diameter of G_{m-1} is at least two, there exists a vertex $A_4 \in V(G_{m-1}) \setminus \mathcal{N}$ that is adjacent to some vertex in \mathcal{N} . If $B \in \mathcal{N}$ such that A_4 is not adjacent to B , then $d(A_4, B) = 2$. By our claim, $\varphi_{m-1}(A_4) \neq \varphi(B)$ and hence $\varphi_{m-1}(A_4) \notin \varphi_{m-1}(\mathcal{N})$. Therefore, φ_{m-1} is not a colouring.

By induction, we may obtain induced endomorphism φ_r of G_r for each r . Furthermore,

$$\varphi(X) = \varphi_{k_1}(X_{k_1}) \vee \varphi_{k_2}(X_{k_2}) \vee \cdots \vee \varphi_{k_s}(X_{k_s}), \quad (8)$$

where $X = X_{k_1} \vee X_{k_2} \vee \cdots \vee X_{k_s} \in V(G_m)$ and $1 \leq \dim(X_{k_i}) = k_i \leq m-1$.

In order to show that φ is an automorphism, it suffices to show that φ is injective. Assume that X and Y are any two distinct vertices in G_m with $d(X, Y) = s$. Thus $\dim(X \cap Y) = m - s$. If $s = 1$, then $\varphi(X) \neq \varphi(Y)$. Now suppose $s \geq 2$. There are 1-dimensional row vectors $X_i, Y_i, i = 1, \dots, s$, such that X, Y can be written as $X = (X \cap Y) \vee X_1 \vee \cdots \vee X_s, Y = (X \cap Y) \vee Y_1 \vee \cdots \vee Y_s$. Let $Z = (X \cap Y) \vee X_1 \vee \cdots \vee X_{s-1} \vee Y_s \in V(G_m)$. By $X \sim Z$, $\dim(\varphi(X) \vee \varphi(Z)) = m + 1$. Applying (8), one has that $\varphi(X) = \varphi_{m-s}(X \cap Y) \vee \varphi_1(X_1) \vee \cdots \vee \varphi_1(X_s), \varphi(Y) = \varphi_{m-s}(X \cap Y) \vee \varphi_1(Y_1) \vee \cdots \vee \varphi_1(Y_s)$ and $\varphi(Z) = \varphi_{m-s}(X \cap Y) \vee \varphi_1(X_1) \vee \cdots \vee \varphi_1(X_{s-1}) \vee \varphi_1(Y_s)$. Therefore, we get $\varphi(X) \vee \varphi(Z) \subseteq \varphi(X) \vee \varphi(Y)$. It follows that $\varphi(X) \neq \varphi(Y)$. Otherwise, one has $\varphi(X) \vee \varphi(Z) \subseteq \varphi(X)$, a contradiction to $\dim(\varphi(X) \vee \varphi(Z)) = m + 1$. Hence, φ is an automorphism, as desired.

By above discussion, $J_q(n, m)$ is a pseudo-core when $n > 2m$. □

Lemma 3.2 *If $n = 2m$, then every Grassmann graph $J_q(n, m)$ is a pseudo-core.*

Proof. Suppose that $n = 2m \geq 4$. For a subspace W of V , the dual subspace W^\perp of W in V is defined by

$$W^\perp = \{v \in V \mid wv^t = 0, \forall w \in W\},$$

where v^\dagger is the transpose of v .

For an endomorphism φ of $J_q(2m, m)$, define the map

$$\varphi^\perp : V(J_q(2m, m)) \longrightarrow V(J_q(2m, m)), \quad A \longmapsto \varphi(A)^\perp.$$

Then φ^\perp is an endomorphism of $J_q(2m, m)$. Note that φ^\perp is an automorphism (resp. colouring) whenever φ is an automorphism (resp. colouring). For any maximal clique \mathcal{M} of $J_q(2m, m)$, $\varphi(\mathcal{M})$ and $\varphi^\perp(\mathcal{M})$ are of different types.

Next we shall show that $J_q(2m, m)$ is a pseudo-core.

Case 1. There exist $[A]_m$ and $\langle X \rangle_m$ such that $[A]_m \cap \langle X \rangle_m \neq \emptyset$ and $\varphi([A]_m)$, $\varphi(\langle X \rangle_m)$ are of the same type.

By Lemma 2.1, the size of $[A]_m \cap \langle X \rangle_m$ is $q + 1$. Then $|\varphi([A]_m) \cap \varphi(\langle X \rangle_m)| \geq q + 1$. Since $\varphi([A]_m)$, $\varphi(\langle X \rangle_m)$ are of the same type, by Lemma 2.2 one gets

$$\varphi([A]_m) = \varphi(\langle X \rangle_m). \quad (9)$$

Note that $A \subseteq X$. Pick any $Y \in \begin{bmatrix} V \\ m+1 \end{bmatrix}$ satisfying $A \subseteq Y$ and $\dim(X \cap Y) = m$. Then $\langle Y \rangle_m \cap [A]_m \neq \emptyset$. By Lemma 2.1 we have $|\varphi(\langle Y \rangle_m) \cap \varphi([A]_m)| \geq q + 1$. By Lemma 2.2 and (9) we obtain either $\varphi(\langle Y \rangle_m) = \varphi(\langle X \rangle_m)$ or $\varphi(\langle Y \rangle_m)$ and $\varphi(\langle X \rangle_m)$ are of different types.

Case 1.1. There exists a $Y \in \begin{bmatrix} V \\ m+1 \end{bmatrix}$ such that $\varphi(\langle Y \rangle_m)$ and $\varphi(\langle X \rangle_m)$ are of different types. For any $B \in \begin{bmatrix} X \cap Y \\ m-1 \end{bmatrix}$, we have that $B \subseteq Y$ and $B \subseteq X$. Since $|[B]_m \cap \langle X \rangle_m| = |[B]_m \cap \langle Y \rangle_m| = q + 1$, we have similarly

$$|\varphi([B]_m) \cap \varphi(\langle X \rangle_m)| \geq q + 1, \quad |\varphi([B]_m) \cap \varphi(\langle Y \rangle_m)| \geq q + 1.$$

Since $\varphi(\langle Y \rangle_m)$ and $\varphi(\langle X \rangle_m)$ are of different types, Lemma 2.2 implies that $\varphi([B]_m) = \varphi(\langle X \rangle_m)$ or $\varphi([B]_m) = \varphi(\langle Y \rangle_m)$ for any $B \in \begin{bmatrix} X \cap Y \\ m-1 \end{bmatrix}$.

Since the size of $\begin{bmatrix} X \cap Y \\ m-1 \end{bmatrix}$ is at least 3, by above discussion, there exist two subspaces $B_1, B_2 \in \begin{bmatrix} X \cap Y \\ m-1 \end{bmatrix}$ such that $\varphi([B_1]_m) = \varphi([B_2]_m)$. Note that $[B_1]_m \cap [B_2]_m \neq \emptyset$ because $X \cap Y \in B_i$ ($i = 1, 2$). If $\varphi([B_1]_m)$ is a star, then φ is a colouring by Lemma 2.7. Suppose $\varphi([B_1]_m)$ is a top. Then $\varphi^\perp([B_1]_m)$ is a star. By Lemma 2.7 again, φ^\perp is a colouring. Hence, φ is also a colouring.

Case 1.2. $\varphi(\langle Y \rangle_m) = \varphi(\langle X \rangle_m)$ for any $Y \in \begin{bmatrix} V \\ m+1 \end{bmatrix}$. Consider a star $[C]_m$ where C satisfies $C \subset X$ and $\dim(C \cap A) = m - 2$. Then $(A \vee C) \subseteq X$ and $\dim(A \vee C) = m$. For any $T \in [C]_m$, since $(A \vee C) \subseteq (A \vee T)$ and $m \leq \dim(A \vee T) \leq m + 1$, there exists a subspace $W \in \begin{bmatrix} V \\ m+1 \end{bmatrix}$ such that $(A \vee T) \subseteq W$ and $\dim(W \cap X) \geq m$ (because $(A \vee C) \subseteq W \cap X$).

Since $T \in \langle W \rangle_m$, $\varphi(T) \in \varphi(\langle W \rangle_m)$. By the condition, $\varphi(\langle W \rangle_m) = \varphi(\langle X \rangle_m)$. Then $\varphi(\langle W \rangle_m) = \varphi([A]_m)$ by (9). It follows that $\varphi(T) \in \varphi([A]_m)$ for all $T \in [C]_m$, and so $\varphi([C]_m) \subseteq \varphi([A]_m)$. Hence, $\varphi([C]_m) = \varphi([A]_m)$. Since $[C]_m \cap [A]_m \neq \emptyset$, similar to the proof of Case 1.1, φ is a colouring.

Case 2. For any two maximal cliques of different types containing common vertices, their images under φ are of different types.

In this case, φ maps the maximal cliques of the same type to the maximal cliques of the same type.

Case 2.1. φ maps stars to stars. In this case φ maps tops to tops by Lemmas 2.1 and 2.2.

If there exist two distinct stars \mathcal{M} and \mathcal{M}' such that $\mathcal{M} \cap \mathcal{M}' \neq \emptyset$ and $\varphi(\mathcal{M}) = \varphi(\mathcal{M}')$, then φ is a colouring by Lemma 2.7. Now suppose $\varphi(\mathcal{M}) \neq \varphi(\mathcal{M}')$ for any two distinct stars \mathcal{M} and \mathcal{M}' with $\mathcal{M} \cap \mathcal{M}' \neq \emptyset$. By Lemma 2.3, φ induces an endomorphism φ_{m-1} of $J_q(2m, m - 1)$ such that $\varphi([A]_m) = [\varphi_{m-1}(A)]_m$. By Lemma 3.1, $J_q(2m, m - 1)$ is a pseudo-core. Thus, φ_{m-1} is an automorphism or a colouring.

We claim that φ_{m-1} is an automorphism of $J_q(2m, m-1)$. For any $C \in \begin{bmatrix} V \\ m \end{bmatrix}$ and $B \in \begin{bmatrix} C \\ m-1 \end{bmatrix}$, since $C \in [B]_m$ and $\varphi([B]_m) = [\varphi_{m-1}(B)]_m$, we have $\varphi(C) \in [\varphi_{m-1}(B)]_m$. Then $\varphi_{m-1}(B) \subseteq \varphi(C)$, which implies that $\varphi_{m-1}(\langle C \rangle_{m-1})$ is a top of $J_q(2m, m-1)$. If $m = 2$, our claim is valid. Now suppose $m \geq 3$ and φ_{m-1} is a colouring. Then $\text{Im}(\varphi_{m-1})$ is a star of $J_q(2m, m-1)$. Note that $\varphi_{m-1}(\langle C \rangle_{m-1}) \subseteq \text{Im}(\varphi_{m-1})$ and $|\varphi_{m-1}(\langle C \rangle_{m-1})| > q+1$, contradicting to Lemma 2.1. Hence, our claim is valid.

Case 2.2. φ maps stars to tops. In this case φ maps tops to stars by Lemmas 2.1 and 2.2.

Note that φ^\perp maps stars to stars. By Case 2.1, φ^\perp is an automorphism. Hence, φ is an automorphism.

By above discussion, we have proved that every Grassmann graph $J_q(2m, m)$ is a pseudo-core. \square

By Lemmas 3.1 and 3.2, we have proved Theorem 1.1.

4 Cores on Grassmann graphs

In this section, we shall show that $J_2(4, 2)$ is not a core and $J_q(n, m)$ is a core under some conditions.

It is well-known (cf. [3, Theorem 6.10 and Corollary 6.2]) that the chromatic number of G satisfies the following inequality:

$$\chi(G) \geq \max \{ \omega(G), |V(G)|/\alpha(G) \}.$$

By [17, Lemma 2.7.2], if G is a vertex-transitive graph, then

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)} \geq \omega(G). \quad (10)$$

Lemma 4.1 *Let G be a Grassmann graph. Then G is a core if and only if $\chi(G) > \omega(G)$. In particular, if $\frac{|V(G)|}{\omega(G)}$ is not an integer, then G is a core.*

Proof. By [7, Corollary 4.2], every distance-transitive graph is core-complete, thus G is core-complete. Then, $\chi(G) > \omega(G)$ implies that G is a core. Conversely, if G is a core, then we must have $\chi(G) > \omega(G)$. Otherwise, there exists an endomorphism f of G such that $f(G)$ is a maximum clique of G , a contradiction to G being a core. Thus, G is a core if and only if $\chi(G) > \omega(G)$.

By [2, p.148, Remark], if the core of G is complete, then $|V(G)| = \omega(G)\alpha(G)$. Assume that $\frac{|V(G)|}{\omega(G)}$ is not an integer. Then $|V(G)| \neq \omega(G)\alpha(G)$. Therefore, the core of G is not complete and hence G is a core. \square

Denote by $\mathbb{F}_q^{m \times n}$ the set of $m \times n$ matrices over \mathbb{F}_q and $\mathbb{F}_q^n = \mathbb{F}_q^{1 \times n}$. Let $G = J_q(n, m)$ where $n > m$. If X is a vertex of G , then $X = [\alpha_1, \dots, \alpha_m]$ is an m -dimensional subspace of the vector space \mathbb{F}_q^n , where

$\{\alpha_1, \dots, \alpha_m\}$ is a basis of X . Thus, X has a matrix representation $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \in \mathbb{F}_q^{m \times n}$ (cf. [9, 18]). For

simplicity, the matrix representation of $X \in V(G)$ is also denoted by X . For matrix representations X, Y of two vertices X and Y , $X \sim Y$ if and only if $\text{rank} \begin{pmatrix} X \\ Y \end{pmatrix} = m+1$. Note that if X is a matrix representation then $X = PX$ (as matrix representation) for any $m \times m$ invertible matrix P over \mathbb{F}_q . Then, $V(G)$ has a matrix representation

$$V(G) = \{X : X \in \mathbb{F}_q^{m \times n}, \text{rank}(X) = m\}.$$

Now, we give an example of Grassmann graph which is not a core as follows.

Example 4.2 *Let $G = J_2(4, 2)$. Then G is not a core. Moreover, $\chi(G) = \omega(G) = 7$ and $\alpha(G) = 5$.*

Proof. Applying the matrix representation of $V(G)$, $G = J_2(4, 2)$ has 35 vertices as follows:

$$\begin{aligned}
A_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
A_5 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, A_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, A_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, A_8 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \\
A_9 &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, A_{10} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, A_{11} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, A_{12} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \\
A_{13} &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, A_{14} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, A_{15} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, A_{16} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \\
A_{17} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{18} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{19} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{20} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \\
A_{21} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, A_{22} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{23} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, A_{24} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \\
A_{25} &= \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, A_{26} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, A_{27} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, A_{28} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
A_{29} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, A_{30} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{31} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{32} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
A_{33} &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{34} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, A_{35} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.
\end{aligned}$$

Suppose that $\mathcal{L}_1 = \{A_1, A_{10}, A_{12}, A_{15}, A_{17}\}$, $\mathcal{L}_2 = \{A_2, A_6, A_{20}, A_{19}, A_{34}\}$, $\mathcal{L}_3 = \{A_3, A_8, A_{21}, A_{22}, A_{35}\}$, $\mathcal{L}_4 = \{A_5, A_9, A_{18}, A_{24}, A_{29}\}$, $\mathcal{L}_5 = \{A_7, A_{14}, A_{23}, A_{27}, A_{33}\}$, $\mathcal{L}_6 = \{A_4, A_{13}, A_{25}, A_{28}, A_{30}\}$, and $\mathcal{L}_7 = \{A_{11}, A_{16}, A_{26}, A_{31}, A_{32}\}$. It is easy to see that $V(G) = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \dots \cup \mathcal{L}_7$ and $\mathcal{L}_1, \dots, \mathcal{L}_7$ are independent sets. Thus $\chi(G) \leq 7$. On the other hand, (10) implies that $\chi(G) \geq \omega(G) = 7$. Therefore, $\chi(G) = \omega(G) = 7$. It follows from Corollary 4.1 that G is not a core. By (10) again, we have $\alpha(G) = 5$. \square

We guess that $J_q(2k, 2)$ ($k \geq 2$) is not a core for all q (which is a power of a prime). But this a difficult problem. Next, we give some examples of Grassmann graph which is a core.

Example 4.3 *If $k \geq 2$, then $J_q(2k+1, 2)$ is core.*

Proof. When $k \geq 2$, let $G = J_q(2k+1, 2)$. Applying (1) and (4) we have

$$\frac{|V(G)|}{\omega(G)} = \frac{q^{2k+1} - 1}{q^2 - 1} = \frac{q^{2k+1} - q}{q^2 - 1} + \frac{1}{q + 1}.$$

Thus $\frac{|V(G)|}{\omega(G)}$ is not an integer for any q (which is a power of a prime). By Lemma 4.1, G is a core. \square

Denote by \mathbb{Z} the integer ring and $\mathbb{Z}[x]$ the polynomial ring in an indeterminate x over \mathbb{Z} . Let $\Phi_t(x)$ be the t th cyclotomic polynomial defined by

$$\Phi_t(x) = \prod_{\substack{1 \leq j \leq t \\ \gcd(j, t) = 1}} (x - \zeta_t^j),$$

where ζ_t is the t th root of unity and $\gcd(j, t)$ is the greatest common divisor of j and t . Recall that $\Phi_t(x)$ is an irreducible polynomial over \mathbb{Z} . The polynomial $x^n - 1$ over \mathbb{Z} has the following factorization into irreducible polynomials over \mathbb{Z} :

$$x^n - 1 = \prod_{j|n} \Phi_j(x). \quad (11)$$

In 1989, Knuth and Wilf gave a factorization of Gaussian binomial coefficient (as a polynomials in $\mathbb{Z}[q]$) (cf. [4, 13]):

$$\begin{bmatrix} n \\ m \end{bmatrix} = \prod_{i=1}^n (\Phi_i(q))^{[n/i] - [m/i] - [(n-m)/i]}, \quad (12)$$

where $\lfloor a \rfloor$ is the largest integer no more than a . Note that $\lfloor n/i \rfloor - \lfloor m/i \rfloor - \lfloor (n-m)/i \rfloor$ is equal to 0 or 1.

Write $G := J_q(n, m)$ (where $4 \leq 2m \leq n$) and $h(q) := \frac{|V(G)|}{\omega(G)} = \frac{\lfloor n \rfloor}{\omega(G)}$, where $h(q)$ is seen as a polynomial in an indeterminate q over the rational number field. By (12) one gets that

$$\omega(G) = \begin{bmatrix} n-m+1 \\ 1 \end{bmatrix} = \prod_{j=1}^{n-m+1} (\Phi_j(q))^{\lfloor (n-m+1)/j \rfloor - \lfloor 1/j \rfloor - \lfloor (n-m)/j \rfloor},$$

$$h(q) = \prod_{j=2}^{n-m+1} (\Phi_j(q))^{\lfloor n/j \rfloor - \lfloor m/j \rfloor - \lfloor (n-m+1)/j \rfloor} \prod_{j=n-m+2}^n (\Phi_j(q))^{\lfloor n/j \rfloor - \lfloor m/j \rfloor - \lfloor (n-m)/j \rfloor}. \quad (13)$$

Theorem 4.4 *Assume that m and $n-m+1$ are not relatively prime. If q (which is a power of a prime) is a sufficiently large integer (i.e., there is a fixed positive integer $c_{n,m}$ such that $q \geq c_{n,m}$), then the Grassmann graph $J_q(n, m)$ is a core.*

Proof. Note that $\lfloor x+y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor$ is equal to 0 or 1 for all real numbers x and y . We have that $\lfloor m/j \rfloor + \lfloor (n-m+1)/j \rfloor$ equals $\lfloor (n+1)/j \rfloor$ or $\lfloor (n+1)/j \rfloor + 1$. Thus,

$$-1 \leq \lfloor n/j \rfloor - \lfloor m/j \rfloor - \lfloor (n-m+1)/j \rfloor \leq 0, \quad j = 2, \dots, n-m+1.$$

Taking the greatest common factor i ($i \geq 2$) of m and $n-m+1$. It is easy to see that

$$\lfloor n/i \rfloor - \lfloor m/i \rfloor - \lfloor (n-m+1)/i \rfloor = -1.$$

Let $f(q) = \prod_{j=2}^{n-m+1} (\Phi_j(q))^{\lfloor n/j \rfloor - \lfloor m/j \rfloor - \lfloor (n-m+1)/j \rfloor}$, $g(q) = \prod_{j=n-m+2}^n (\Phi_j(q))^{\lfloor n/j \rfloor - \lfloor m/j \rfloor - \lfloor (n-m)/j \rfloor}$. Then $f(q)$, $g(q)$ are monic polynomials in $\mathbb{Z}[q]$ and $\deg(g(q)) \geq 1$ because $\Phi_i(q)$ is a factor of $g(q)$. By (13), we have $h(q) = f(q)/g(q)$. Recall that $\Phi_j(q)$, $j = 1, \dots, n$, are irreducible polynomials in $\mathbb{Z}[q]$. We have $g(q) \nmid f(q)$. By the polynomial division algorithm, $f(q) = g(q)f_1(q) + r(q)$, where $f_1(q), r(q) \in \mathbb{Z}[q]$, $r(q) \neq 0$ and $\deg(r(q)) < \deg(g(q))$. Thus, $h(q) = f_1(q) + r(q)/g(q)$. Clearly, if q is a sufficiently large integer (i.e., there is a fixed positive integer $c_{n,m}$ such that $q \geq c_{n,m}$), then $h(q)$ is not an integer. Thus, Lemma 4.1 implies that $J_q(n, m)$ is a core if q is a sufficiently large integer. \square

When m and $n-m+1$ are not relatively prime, we guess that $J_q(n, m)$ is a core for all q (which is a power of a prime).

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